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DEPARTMENT OF MATHEMATICS

MMAT5510 Foundation of Advanced Mathematics 2017-2018
Supplementary Exercise 5

1. (a) Let A be a subset of $\mathbb{R}$. State the definition of a cluster point of A.

   (b) Let A be a subset of $\mathbb{R}$, c be a cluster point of A, and $f : A \to \mathbb{R}$ be a function.
   State the definition of $\lim_{x\to c} f(x) = L$, where L is a real number.
   (Remark: The definition is usually called δ - ϵ definition.)

   (c) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = 3x + 2$.
   By using the definition stated in (b), show that $\lim_{x\to 1} f(x) = 5$.

   Ans:

   (a) c is said to be a cluster point of A if
   for all $\delta > 0$, there exists $x \in A\backslash\{c\}$ such that $|x - c| < \delta$.
   Symbolic: $(\forall\delta > 0)(\exists x \in A\backslash\{c\})(|x - c| < \delta)$

   (b) $\lim_{x\to c} f(x) = L$ if
   for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ with $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.
   Symbolic: $(\forall\epsilon > 0)(\exists\delta > 0)(\forall x \in A \text{ and } 0 < |x - c| < \delta)(|f(x) - L| < \epsilon)$

   (c) Let $\epsilon > 0$. Take $\delta = \frac{\epsilon}{3} > 0$.
   Then, for all $0 < |x - 1| < \delta$, we have

$$\begin{aligned} |x-1| &< \delta \\ |x-1| &< \frac{\epsilon}{3} \\ |3x-3| &< \epsilon \\ |(3x+2)-5| &< \epsilon \\ |f(x)-5| &< \epsilon \end{aligned}$$

   Therefore, $\lim_{x\to 1} f(x) = 5$.

2. Using the δ - ϵ definition, show that

   (a) $\lim_{x\to c} x^3 = c^3$;

   (b) $\lim_{x\to 2} \frac{x^2 - 5x + 6}{x - 2} = -1$.

   Ans:

   (a) Let $\epsilon > 0$. Take $\delta = \min\{1, \frac{\epsilon}{3|c|^2 + 3|c| + 1}\}$.
   Then, for all $0 < |x - c| < \delta$, we have

$$\begin{aligned} &|x-c| < 1 \\ -|c|-1 \leq c-1 < &\; x < c+1 \leq |c|+1 \\ &|x| < |c|+1 \end{aligned}$$

Also,

$$|x^2 + xc + c^2| \leq |x|^2 + |x||c| + |c|^2 \leq (|c|+1)^2 + (|c|+1)|c| + |c|^2 = 3|c|^2 + 3|c| + 1$$

Then,

$$\begin{aligned} |x-c||x^2+xc+c^2| &< \frac{\epsilon}{3|c|^2+3|c|+1} \cdot (3|c|^2+3|c|+1) \\ |x^3 - c^3| &< \epsilon \\ |f(x) - c^3| &< \epsilon \end{aligned}$$

Therefore, $\lim_{x \to c} f(x) = c^3$.

(b) Let $\epsilon > 0$. Take $\delta = \epsilon$.

Then, for all $0 < |x-2| < \delta$, we have

$$\begin{aligned} |x-2| &< \epsilon \\ |(x-3) - (-1)| &< \epsilon \\ \left| \frac{(x-3)(x-2)}{x-2} - (-1) \right| &< \epsilon \qquad (0 < |x-2| \Rightarrow x - 2 \neq 0) \\ |f(x) - (-1)| &< \epsilon \end{aligned}$$

Therefore, $\lim_{x \to 2} f(x) = -1$.

3. Let A be a subset of $\mathbb{R}$, c be a cluster point of A, and $f, g : A \to \mathbb{R}$ be functions such that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$.
Show that

(a) $\lim_{x \to c} f(x) + g(x) = L + M$;

(b) if $g(x) \neq 0$ for all $x \in A$ and $M \neq 0$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Ans:

(a) Let $\epsilon > 0$.

Since $\lim_{x \to c} f(x) = L$, there exists $\delta_1 > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta_1$, we have $|f(x) - L| < \frac{\epsilon}{2}$,
and $\lim_{x \to c} g(x) = M$, there exists $\delta_2 > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta_2$, we have $|g(x) - M| < \frac{\epsilon}{2}$.

Take $\delta = \min\{\delta_1, \delta_2\} > 0$, then for all $x \in A$ with $0 < |x - c| < \delta$, we have

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &\leq |f(x) - L| + |g(x) - M| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Therefore, $\lim_{x \to c} f(x) + g(x) = L + M$.

(b) Since $\lim_{x\to c} g(x) = M$, consider $\epsilon_0 = \frac{|M|}{2}$, there exists $\delta_1 > 0$ such that for all $x \in A$ with $0 < |x-c| < \delta_1$, we have $|g(x) - M| < \epsilon_0 = \frac{|M|}{2}$ and then

$$M - \frac{|M|}{2} < g(x) < M + \frac{|M|}{2}.$$

If $M > 0$, then $|M| = M$ and $0 < \frac{M}{2} < g(x)$, so $\frac{M}{2} < |g(x)|$.

If $M < 0$, then $|M| = -M$ and $g(x) < M + \frac{|M|}{2} = -\frac{|M|}{2}$, so $\frac{M}{2} < |g(x)|$.

Therefore, for all $x \in A$ with $0 < |x-c| < \delta_1$, we have $\frac{M}{2} < |g(x)|$ and so $\frac{1}{|g(x)|} < \frac{2}{|M|}$.

Let $\epsilon > 0$.

Since $\lim_{x\to c} f(x) = L$, there exists $\delta_2 > 0$ such that for all $x \in A$ with $0 < |x-c| < \delta_2$, we have $|f(x) - L| < \frac{|M|\epsilon}{4}$,

and $\lim_{x\to c} g(x) = M$, there exists $\delta_2 > 0$ such that for all $x \in A$ with $0 < |x-c| < \delta_3$, we have $|g(x) - M| < \frac{|M|^2\epsilon}{4|L|}$.

Take $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0$, then for all $x \in A$ with $0 < |x-c| < \delta$, we have

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| &= \left|\frac{f(x)M - g(x)L}{g(x)M}\right| \\
&= \left|\frac{f(x)M - LM + LM - g(x)L}{g(x)M}\right| \\
&= \left|\frac{(f(x)-L)M - (g(x)-M)L}{g(x)M}\right| \\
&\leq \frac{|f(x)-L|}{|g(x)|} + \frac{|g(x)-M||L|}{|g(x)||M|} \\
&< \frac{|M|\epsilon}{4} \cdot \frac{2}{M} + \frac{|M|^2\epsilon}{4|L|} \cdot \frac{2}{|M|} \cdot \frac{|L|}{|M|} \\
&\leq \epsilon
\end{aligned}$$

Therefore, $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$.

4. Suppose that c is a cluster point of A and $f, g : A \to \mathbb{R}$ are two functions such that $\lim_{x\to c} f(x) = 0$ and g is bounded on a neighborhood of c.

   Show that $\lim_{x\to c} f(x)g(x) = 0$.

   Ans:

   Since g is bounded on a neighborhood of c, there exist $\delta_1 > 0$ and $M > 0$ such that $|g(x)| \leq M$ for all $x \in A$ with $0 < |x-c| < \delta_1$.

   Let $\epsilon > 0$.

   Since $\lim_{x\to c} f(x) = 0$, there exists $\delta_2 > 0$ such that for all $x \in A$ with $0 < |x-c| < \delta_2$, we have $|f(x) - 0| < \frac{\epsilon}{M}$, i.e. $|f(x)| < \frac{\epsilon}{M}$.

Take $\delta = \min\{\delta_1, \delta_2\} > 0$, then for all $x \in A$ with $0 < |x - c| < \delta$, we have

$$\begin{aligned} |f(x)g(x) - 0| &= |f(x)||g(x)| \\ &< \frac{\epsilon}{M} \cdot M \\ &= \epsilon \end{aligned}$$

Therefore, $\lim_{x \to c} f(x)g(x) = 0$.

5. Let A be a subset of $\mathbb{R}$, c be a cluster point of A, and $f : A \to \mathbb{R}$ be a function such that $a \leq f(x) \leq b$ for all $x \in A \backslash \{c\}$ and $\lim_{x \to c} f(x) = L$ exists.

   Show that $a \leq L \leq b$.

   Ans:

   Let $\epsilon > 0$.

   Since $\lim_{x \to c} f(x) = L$ exists, there exists $\delta > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$ and so

$$a - \epsilon \leq f(x) - \epsilon < L < f(x) + \epsilon \leq b + \epsilon.$$

   We have $a - \epsilon \leq L \leq b + \epsilon$ for all $\epsilon > 0$ and so $a \leq L \leq b$.

6. Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \backslash \{c\} \to \mathbb{R}$.

   The right hand limit of f at c is $L \in \mathbb{R}$ (denoted by $\lim_{x \to c^+} f(x) = L$) if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $c - \delta < x < c$, we have $|f(x) - L| < \epsilon$.

   The left hand limit (denoted by $\lim_{x \to c^-} f(x) = L$) can be defined in a similar way.

   Show that $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$.

   Ans:

   ($\Rightarrow$) Suppose that $\lim_{x \to c} f(x) = L$.

   Let $\epsilon > 0$, there exist $\delta > 0$, such that for all $0 < |x - c| < \delta$ (i.e. $c - \delta < x < c$ or $c < x < c + \delta$), we have $|f(x) - L| < \epsilon$.

   In particular, when $c < x < c + \delta$, we have $|f(x) - L| < \epsilon$ and so $\lim_{x \to c^+} f(x) = L$; when $c - \delta < x < c$, we also have $|f(x) - L| < \epsilon$ and so $\lim_{x \to c^-} f(x) = L$.

   ($\Leftarrow$) Suppose that $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$.

   Let $\epsilon > 0$.

   Since $\lim_{x \to c^+} f(x) = L$, there exist $\delta_1 > 0$, such that for all $c < x < c + \delta_1$, we have $|f(x) - L| < \epsilon$.

   Since $\lim_{x \to c^-} f(x) = L$, there exist $\delta_2 > 0$, such that for all $c - \delta_2 < x < c$, we have $|f(x) - L| < \epsilon$.

   Take $\delta = \min\{\delta_1, \delta_2\} > 0$.

   Then for all $c < x < c + \delta$, we have $c < x < c + \delta_1$, and so $|f(x) - L| < \epsilon$; for all $c - \delta < x < c$, we have $c - \delta_2 < x < c$, and so $|f(x) - L| < \epsilon$.

   That is, for all $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$, so $\lim_{x \to c} f(x) = L$.